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On the spectra of Riemann–Liouville fractional Brownian motion

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Abstract. We study the spectrum of the fractional Brownian motion of Riemann–Liouville type using two approaches, namely the double frequency spectral density and the Wigner–Ville spectrum. The bifrequency representation gives a complex-valued function which contains two distinctive terms. These terms can be identified as the diagonal and off-diagonal distribution of the spectrum in a frequency–frequency plane. The physical interpretation of these two terms is briefly discussed. A calculation of Wigner–Ville spectrum gives an alternative way of representing the spectrum of this nonstationary process in the time–frequency plane. Asymptotic approximation of the Wigner–Ville spectrum is obtained. We show that the large-time average spectrum of Riemann–Liouville fractional Brownian motion exhibits a power law in the high-frequency range.

1. Introduction

Many natural phenomena and man-made processes [1] exhibit empirical spectra which obey the fractional power law of the form $1/f^{\alpha}$, $1 < \alpha < 3$. this ubiquitous feature is frequently observed, for example, in mountain profiles [2], electrical noise in semiconductor devices [3], Kolmogorov's 5/3 power law in turbulence [4], etc. However, a universally acceptable representation and characterization of the $1/f^{\alpha}$ spectral behaviour is still unknown [5, 6]. Perhaps the only similarity among these systems is the mathematical description that leads to such spectra. Several useful models are available, among them a class of processes based on the fractional integral representation. The most widely used model from this class is that of fractional Brownian motion introduced by Mandelbrot and Van Ness [7]. Fractional Brownian motion is in itself not a stationary process, but its increments are. This property, together with self-affinity, allows one to obtain a generalized spectral density of power-law type for the process itself.

Since fractional Brownian motion (FBM) of Mandelbrot and Van Ness (MV) is defined for all times, it is not suitable for modelling phenomena that occur only in positive time. For the latter purpose it may be useful to consider the one-sided FBM. In fact, such a process was used by Barnes and Allan [8] to model flicker noise. Unfortunately, the undue emphasis on the time origin in the definition of this process has resulted in non-stationary increments, which in turn leads to the failure of its spectral density to obey the power law.

The main aim of this paper is to reconsider the one-sided FBM (which we shall call Riemann–Liouville fractional Brownian motion (RL FBM) in this paper) to study the possibility of extracting more information from its spectral density. We shall investigate two types of generalized spectral densities for this non-stationary process, namely the double-frequency spectral representation and the Wigner–Ville spectrum. In particular, we want to find the conditions under which the spectrum of RL FBM exhibits power-law behaviour.

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2. Riemann–Liouville fractional Brownian motion

In an attempt to find a representation of a Gaussian $1/f$ process, Barnes and Allan [8] defined a process based on the derivative of $(H+\frac{1}{2})$th order Riemann–Liouville fractional integral [9]

$$X_H(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\int_0^t (t-\tau)^{H-\frac{1}{2}}\eta(\tau)\,\mathrm{d}\tau \tag{1}$$

where $H > 0$ and $h(t)$, $t > 0$ is the one-sided Gaussian white noise with mean zero and covariance

$$\langle\eta(t)\eta(s)\rangle = \theta(t)\,\theta(s)\,\delta(t-s) \tag{2}$$

where $\theta(t)$ is the unit step function. $X_H(t)$ is Gaussian process with mean zero and covariance given by

$$R_{X_H}(t,s) \equiv \langle X_H(t)X_H(s)\rangle = \frac{t^{H+\frac{1}{2}}s^{H-\frac{1}{2}}}{\left(H+\frac{1}{2}\right)\Gamma\left(H+\frac{1}{2}\right)^2}\,{}_2F_1\left(\frac{1}{2}-H, 1, H+\frac{1}{2}, \frac{t}{s}\right) \tag{3}$$

for $s > t$ and ${}_2F_1$ is the hypergeometric function. We remark that $X_H(t)$ can also be defined for $-\frac{1}{2} \leqslant H \leqslant 0$, provided it is regarded as a generalized process, and terms involving fractional powers of time are to be regarded as a distribution [10].

Since RL FBM is defined for $t \geqslant 0$, it is not time-translation invariant and therefore is a non-stationary process. It can be easily shown, however, that $X_H(t)$ is statistically self-affine as it satisfies the following scaling relation

$$X_H(t) \equiv a^{-H}X_H(at) \qquad a > 0 \tag{4}$$

where $\equiv$ denotes equality in the statistical sense. The increments of RL FBM are not stationary and therefore not self-affine. We remark that self-affinity of the process together with stationary increments is crucial for the spectrum of the process to obey a power law.

Before we discuss MV FBM, let us recall that the Ornstein–Uhlenbeck process [11] starting at $t = 0$ is defined (up to a constant) by replacing $(t-\tau)^{H-\frac{1}{2}}$ by $\exp[-|t-\tau|]$ in (1). This non-stationary Gaussian process becomes stationary if the lower limit of the integral is changed from zero to $-\infty$. If a similar change is carried out for RL FBM, i.e. one uses the Weyl fractional integral [9] instead of the Riemann–Liouville fractional integral, we have

$$X_H(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\int_{-\infty}^t (t-\tau)^{H-\frac{1}{2}}\eta(t)\,\mathrm{d}\tau \tag{5}$$

with $\eta(t)$ defined for all t. Now (5) is divergent for $H > 0$, but $X_H(t)$ is defined as a generalized stationary process for $-\frac{1}{2} < H < 0$. In their attempt to overcome this problem, Mandelbrot and Van Ness introduced the reduced form of FBM for $0 < H < 1$:

$$B_H(t) \equiv X_H(t) - X_H(0) = \frac{1}{\Gamma\left(H+\frac{1}{2}\right)} \times \left[\int_{-\infty}^0 \left[|t-\tau|^{H-\frac{1}{2}} - |\tau|^{H-\frac{1}{2}}\right]\eta(\tau)d\tau + \int_0^t |t-\tau|^{H-\frac{1}{2}}\,\eta(\tau)\,\mathrm{d}\tau\right]. \tag{6}$$

It is known that (6) converges and one obtains $B_H(t)$ as a non-stationary self-affine process with stationary increments. In fact, it can be shown that $B_H(t)$ defined above is the only mean-zero, mean-square continuous Gaussian self-affine process (satisfying $B_H(0) = 0$) with stationary increments [12]. This last property allows one to obtain a generalized spectral density that obeys a power law behaviour.

Despite the nice properties of MV FBM [13], it cannot account for an important $1/f$ a process, namely the flicker noise with $\alpha = 1$ or $H = 0$. In this case, FBM degenerates to the trivial process $B_H(t) = 0$. More generally, $H < 0$ in (6) leads to processes that are not mean-square continuous; while $H > 1$ gives processes whose increments are not stationary. In comparison, RL FBM cover a larger range of H (with $-\frac{1}{2} < H < 0$ as a generalized process), which may allow one to model some phenomena with properties not covered by MV FBM. We shall give a preliminary investigation of this possibility by studying the properties of the generalized spectral density of the process.

3. Bifrequency spectrum of Riemann–Liouville fractional Brownian motion

The spectral density of a stationary process is usually obtained by taking the Fourier transform of its time translation invariant covariance via the Wiener–Khintchine theorem [14]. This procedure does not apply to non-stationary processes, since in this case the covariance depends on both time parameters explicitly, which consequently leads to spectral density not dependent on a single frequency variable, ω or $2\pi f$. Instead, one should consider coordinates in the plane of two frequencies (ω_1 and ω_2). In the case of a stationary process, the spectral density lies on the bisector ($\omega_1 = \omega_2 = \omega$) .

A natural extension of the notion of spectral density to non-stationary process is to apply the 'double ' Fourier transform to the covariance :

$$S_{X_H}(\omega_1, \omega_2) = \iint R_{X_H}(t_1, t_2) \mathrm{e}^{\mathrm{i}(\omega_1 t_1 - \omega_2 t_2)} \, \mathrm{d}t_1 \mathrm{d}t_2 \tag{7}$$

where $S(\omega_1, \omega_2)$ denotes the bifrequency spectrum [15]. Note that (7) is exactly the double Fourier transform of $R_{X_H}(t_1, t_2)$ but the exponential terms carry opposite signs. Direct evaluation of this integral with the substitution of $R_{X_H}(t_1, t_2)$ is rather difficult. Instead, we make use of the fact that RL FBM can be written in the following convolution form:

$$X_H(t) = h(t)_* \eta(t) \tag{8a}$$

with

$$h(t) = \begin{cases} \dfrac{t^{H-\frac{1}{2}}}{\Gamma(\lambda)} & t > 0 \\ 0 & t \leqslant 0 \end{cases} \tag{8b}$$

where $\lambda \equiv H + \frac{1}{2}$. The Fourier transform of $h(t)$ is given by

$$h(\omega) = \frac{\mathrm{e}^{\mathrm{i}\lambda\pi/2}}{\Gamma(\lambda)(\omega + \mathrm{i}o)^{\lambda}}. \tag{9}$$

The 'double' Fourier transform of the covariance of the one-sided white noise is

$$\langle \eta^*(\omega_1)\eta(\omega_2) \rangle = \frac{-\mathrm{i}}{(\omega_2 - \omega_1 - \mathrm{i}o)} = \pi\delta(\omega_2 - \omega_1) - \frac{\mathrm{i}}{(\omega_2 - \omega_1)}. \tag{10}$$

Combining (9) and (10), one gets the following complex-valued bifrequency spectrum of RL FBM:

$$S_{X_H}(\omega_1, \omega_2) = \langle X_H^*(\omega_1)X_H(\omega_2)\rangle = h^*(\omega_1)h(\omega_2)\langle\eta^*(\omega_1)\eta(\omega_2)\rangle$$
$$= \frac{1}{2[(\omega_1 + io)(\omega_2 - io)]^\lambda}\left[\delta(\omega_2 - \omega_1) - \frac{\mathrm{i}}{\pi(\omega_2 - \omega_1)}\right]. \qquad (11)$$

One can interpret $S_{X_H}(\omega_1, \omega_2)$ as 'generalized energy', in the following sense. Double-frequency spectrum allows one to know how the spectral 'mass' of the process $X_H(t)$ is distributed in the (ω_1, ω_2)-plane. The first term in (11) is the time-translational invariant or the stationary part of the bifrequency spectrum. It contains only the mean-square value of the random amplitude, i.e. the 'amount of energy'. The second term represents the non-stationary part of the bifrequency spectrum. It contains information about both the energy and the correlation between amplitudes with unequal 'frequencies' ω_1 and ω_2, i.e. the 'energy spread' about the diagonal $\omega_1 = \omega_2$. In other words, the generalized energy of the process is not localized in frequency. The amount of the spread of energy tells us the degree of deviation from stationarity. In the case of positive ω_1 and ω_2, we note that $S_{X_H}(\omega_1, \omega_2) = S^*_{X_H}(\omega_1, \omega_2)$, which means a point located symmetrically in the bisector $\omega_1 = \omega_2$; the values of the bispectral density are complex conjugates of one another, but are real on the bisector itself.

It is also interesting to note that if we write $W = \omega_2 - \omega_1$ and keep either ω_1 or ω_2 constant, then the terms in the square bracket in (11) form a dispersion relation with the real and imaginary parts, forming a pair of Hilbert transforms. This is just a consequence of the fact that Fourier transform of a step function gives rise to a dispersion relation. However, we remark that $S_{X_H}(\omega_1, \omega_2)$ does not form a dispersion relation. Finally, it should be mentioned that the double frequency spectral density given above is slightly different from the result obtained by Wyss [16] using direct double Fourier transform without considering complex conjugation in one of the transforms. The double-frequency spectrum obtained here allows a more transparent interpretation for the energy distribution, especially in the positive frequencies domain.

4. Wigner–Ville Spectrum

Instead of the bifrequency spectrum, one can also investigate the spectral behaviour of a non-stationary process using a time-varying spectrum. An important class of time-dependent spectral representation is the Wigner–Ville spectrum [17]. It was first introduced as the Wigner distribution in quantum mechanics [18] and later applied to signal analysis by Ville [19]. For a real-valued non-stationary Gaussian process with covariance $R_X(t, s)$ the Wigner–Ville spectrum is given by [15]

$$W_X(t, \omega) = \int_{-\infty}^{\infty} R_X\left(t - \frac{\tau}{2}, t + \frac{\tau}{2}\right) \mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau. \qquad (12)$$

For a stationary Gaussian process, Wigner–Ville spectrum reduces to the conventional spectrum. The main disadvantage of this spectrum is that it is not always positive. Despite this shortcoming, the Wigner–Ville spectrum is still regarded as one of the most useful time-frequency spectral representations.

For RL FBM, a direct substitution of the covariance $R_{X_H}(t_1, t_2)$ in (12) once again gives a rather complicated integral. A simpler way of calculating $W_{X_H}(t, \omega)$ in this case is to make use of the property of the Wigner–Ville spectrum for convoluted signals. Recall that RL FBM can be expressed in the convolution form $X_H(t) = h(t)_* \eta(t)$. If W_h and W_η are the Wigner–Ville spectra of $h(t)$ and $\eta(t)$, respectively, then one can show that [17]

$$W_{X_H}(t,\omega) = \int_{-\infty}^{\infty} W_h(t-\tau,\omega)\, W_\eta(\tau,\omega)\,\mathrm{d}\tau. \tag{13}$$

In the case of the one-sided white noise $\eta(t)$, one gets

$$W_\eta(t,\omega) = \int_{-\infty}^{\infty} \theta\left(t-\frac{\tau}{2}\right)\theta\left(t+\frac{\tau}{2}\right)\delta(\tau)\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau = \int_{-2t}^{2t} \delta(t)\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau = \theta(t). \tag{14}$$

If we let $g(\tau) \equiv h(t-\frac{\tau}{2})\,h(t+\frac{\tau}{2})$ then

$$g(\tau) = \begin{cases} \Gamma(\lambda)^{-2}(t^2 - \frac{\tau^2}{4}) & -2t < \tau < 2t \\ 0 & \text{otherwise}\,. \end{cases}$$

One gets

$$W_h(t,\omega) = \int_{-2t}^{2t} g(\tau)\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau = \theta(t)\frac{2\sqrt{\pi}\,t^H}{\Gamma(\lambda)\omega^H} J_H(2\omega t) \tag{15}$$

where J_H is the Bessel function of first kind of order H [20]. Finally we have

$$\begin{aligned} W_{X_H}(t,\omega) &= \frac{2\sqrt{\pi}}{\Gamma(\lambda)\omega^H}\int_{-\infty}^{\infty}\theta(\tau)\theta(t-\tau)(t-\tau)^H J_H[2\omega(t-\tau)]\,\mathrm{d}\tau \\ &= \frac{2\sqrt{\pi}}{\Gamma(\lambda)\omega^H}\int_0^t (t-\tau)^H J_H[2\omega(t-\tau)]\,\mathrm{d}\tau. \end{aligned} \tag{16}$$

It follows that

$$W_{X_H}(t,\omega) = \frac{\pi\omega t}{\omega^{2H+1}}\left[J_H(2\omega t)\boldsymbol{H}_{H-1}(2\omega t) - J_{H-1}(2\omega t)\boldsymbol{H}_H(2\omega t)\right] \tag{17}$$

where $\boldsymbol{H}_H$ is the Struve function of order H. One can verify the scaling property of this spectrum by considering a scaled FBM defined as $X_{H,a}(t) = X_H(at)$, $a > 0$, and easily show that it satisfies

$$W_{X_{H,a}}(t,\omega) = \frac{1}{a}W_{X_H}\left(at, \frac{\omega}{a}\right). \tag{18}$$

It follows from (17) that

$$W_{X_{H,a}}(t,\omega) = W_{a^H X_H}(t,\omega) \tag{19}$$

which is a second-order manifestation RL FBM's self-affinity in terms of rescaled time-frequency spectrum. Note that for $H = \frac{1}{2}$, equation (17) reduces to the Wigner–Ville spectrum for the ordinary Brownian motion:

$$\begin{aligned} W_{B_{1/2}}(t,\omega) &= \frac{\pi}{\omega^2}(\omega t)\left[\sqrt{\frac{1}{\pi\omega t}}\sin(2\omega t)\sqrt{\frac{1}{\pi\omega t}}\sin(2\omega t)\right. \\ &\qquad \left. -\sqrt{\frac{1}{\pi\omega t}}\cos(2\omega t)\sqrt{\frac{1}{\pi\omega t}}\{1-\cos(2\omega t)\}\right] \\ &= \frac{1}{\omega^2}[1-\cos(2\omega t)] \end{aligned} \tag{20}$$

which agrees with the result obtained by Flandrin [21] as a special case of the Wigner–Ville spectrum for MV FBM.

In general (except for half-integer H) $W_{X_H}(t,\omega)$ does not exist in any closed form. Both J_H and $\boldsymbol{H}_H$ can be expressed as infinite series or transformed into special functions of other type such as the generalized hypergeometric functions. In order to extract some information (for example, explicit time and frequency dependence) from $W_{X_H}(t,\omega)$, we shall consider the asymptotic approximation for $wt \gg 1$. The series expansions for $J_H(z)$ and $\boldsymbol{H}_H(z)$ are [22]

$$J_H(z) \approx \sqrt{\frac{2}{\pi z}}\left[\cos\left(z - \frac{\pi H}{2} - \frac{\pi}{4}\right)\sum_{m=0}^{\infty}\frac{(-1)^m(H,2m)}{(2z)^{2m}} - \sin\left(z - \frac{\pi H}{2} - \frac{\pi}{4}\right)\right.$$
$$\left.\times\sum_{m=0}^{\infty}\frac{(-1)^m(H,2m+1)}{(2z)^{2m+1}}\right] \tag{21}$$

$$\boldsymbol{H}_H(z) \approx Y_H(z) + \frac{1}{\pi}\sum_{m=0}^{p-1}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(H+\frac{1}{2}-m\right)}\frac{1}{(z/2)^{2m-H+1}} \tag{22}$$

where $Y_H(z)$ is an Hth order Bessel function of second kind with the asymptotic expansion given by

$$Y_H(z) \approx \sqrt{\frac{2}{\pi z}}\left[\sin\left(z - \frac{\pi H}{2} - \frac{\pi}{4}\right)\sum_{m=0}^{\infty}\frac{(-1)^m(H,2m)}{(2z)^{2m}}\right.$$
$$\left.- \cos\left(z - \frac{\pi H}{2} - \frac{\pi}{4}\right)\sum_{m=0}^{\infty}\frac{(-1)^m(H,2m+1)}{(2z)^{2m+1}}\right] \tag{23}$$

and (H,m) is the Hankel notation:

$$(H,m) \equiv \frac{\Gamma\left(H+m+\frac{1}{2}\right)}{m!\,\Gamma\left(H-m+\frac{1}{2}\right)}. \tag{24}$$

In the asymptotic expansion of (22), the remainder after p terms is of the same sign as, but numerically less than, the first term neglected, provided $(p+\frac{1}{2}-H) \geqslant 0$ [22]. If we expand the series up to $O(1/z)$, i.e. to $p = 1$, then we need to impose the restriction $H < \frac{3}{2}$. The asymptotic expansion of (17) up to $O(1/\omega t)$ as $\omega t \to \infty$ is given by

$$W_{X_H}(t,\omega) \approx \frac{1}{\omega^{2H+1}}\left[1 + \frac{\sin(2\omega t - \frac{\pi H}{2} - \frac{\pi}{4})}{\Gamma\left(H+\frac{1}{2}\right)(\omega t)^{\frac{1}{2}-H}} + \frac{\cos\left(2\omega t - \frac{\pi H}{2} - \frac{\pi}{4}\right)}{\Gamma\left(H-\frac{1}{2}\right)(\omega t)^{\frac{3}{2}-H}}\right]$$
$$\approx \frac{1}{\omega^{2H+1}}\left[1 + \frac{\sin(2\omega t - \frac{\pi H}{2} - \frac{\pi}{4})}{\Gamma\left(H+\frac{1}{2}\right)(\omega t)^{\frac{1}{2}-H}}\right] \qquad \omega t \gg 1 \tag{25}$$

for $-\frac{1}{2} < H < \frac{3}{2}$; the third term is neglected as compared with the first two. One readily notices that the large argument expansion of the special functions allows one to freely set the range of the temporal evolution and the frequency such that $\omega t \gg 1$. Nevertheless, one should bear in mind that the upper bound of the frequencies depends on the sampling rate (Δt) or is given by the Nyquist frequency, i.e. $\omega_c = \pi/\Delta t$. In addition, there also exists a low-frequency cut-off ω_0 which arises owing to the finite observation time. Therefore, it seems appropriate to consider $\omega_0 \ll \omega < \omega_c$ together with large-time approximation as the latter condition always prevails in most occurances of $1/f$ noises [8].

It has been mentioned earlier that the Wigner–Ville spectrum is not always positive, while (25) turns out to be positive for $H < \frac{1}{2}$. However, for $\frac{1}{2} < H < \frac{3}{2}$, $W_{X_H}(t,\omega)$ oscillates between increasing values of positive and negative amplitudes.

In order to obtain explicit frequency dependence of the asymptotic Wigner–Ville spectrum, we consider the time-average of $W_{X_H}(t,\omega)$, thus smoothening out its temporal dependence. This is done by averaging over a time interval of length T, namely

$$\begin{aligned}\overline{W_{X_H}}(T,\omega) &= \frac{1}{T}\int_0^T W_{X_H}(t,\omega)\,\mathrm{d}t \\ &= \frac{1}{T}\int_0^T \left[1 + \frac{\sin\left(2\omega t - \frac{\pi H}{2} - \frac{\pi}{4}\right)}{\Gamma\left(H+\frac{1}{2}\right)(\omega t)^{\frac{1}{2}-H}}\right]\frac{1}{\omega^{2H+1}}\,\mathrm{d}t \\ &= \frac{1}{\omega^{2H+1}}\left[1 + \frac{2^{\frac{1}{2}-H}}{\Gamma\left(H+\frac{1}{2}\right)(2\omega T)}\left\{\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(H+\frac{1}{2})}(\mathrm{i})^{\frac{1}{2}-H}\gamma\left(H+\frac{1}{2}, \mathrm{i}2\omega T\right)\right.\right. \\ &\quad \left.\left. +\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(H+\frac{1}{2})}(-\mathrm{i})^{\frac{1}{2}-H}\gamma(H+\tfrac{1}{2}, -\mathrm{i}2\omega T)\right\}\right] \end{aligned} \tag{26}$$

where $\gamma(H+\frac{1}{2}, \pm 2\mathrm{i}\omega T)$ is an incomplete gamma function. The spectrum in the above form does not allow one to deduce directly the large-time average of $W_{X_H}(t,\omega)$. However, one can show that for large T the second term in (25) vanishes or equals a finite constant. Consider the following limit:

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\frac{1}{\Gamma\left(H+\frac{1}{2}\right)}&\int_a^{T+a}\frac{1}{(\omega t)^{\frac{1}{2}-H}}\sin\left(2\omega t - \frac{\pi H}{2} - \frac{\pi}{4}\right)\mathrm{d}t \\ &= \lim_{T\to\infty}\frac{1}{T}\frac{1}{\Gamma\left(H+\frac{1}{2}\right)}\left[\int_a^{T+a}(\omega t)^{H-\frac{1}{2}}\left\{\cos\left(\frac{\pi}{2}\left[H+\tfrac{1}{2}\right]\right)\sin(2\omega t)\right.\right. \\ &\quad \left.\left. -\sin\left(\frac{\pi}{2}\left[H+\tfrac{1}{2}\right]\right)\cos(2\omega t)\right\}\mathrm{d}t\right]. \end{aligned} \tag{27}$$

According to the theory of improper semi-convergent Lebesgue integrals (see, e.g. [23]), the following improper trigonometric integrals are implied by the Abel theorem:

$$\lim_{T\to\infty}\int_a^{T+a} f(t)\cos\omega t\,\mathrm{d}t \quad \text{and} \quad \lim_{T\to\infty}\int_a^{T+a} f(t)\sin\omega t\,\mathrm{d}t \qquad a>0 \tag{28}$$

converge if $f(t) \geqslant 0$ and decreasing with $f(t) \to 0$ as $t \to \infty$. These conditions are satisfied by $f(t) = \text{constant}\,(\omega t)^{H-\frac{1}{2}}$ for $H \leqslant \frac{1}{2}$. With the inclusion of the averaging factor $1/T$, we conclude that the limit (27) converges for $H < \frac{3}{2}$. Hence, the large-time average of the asymptotic Wigner–Ville spectrum is given by

$$\overline{W_{X_H}}(\omega) = \lim_{T\to\infty}\overline{W_{X_H}}(T,\omega) \sim \frac{1}{\omega^{2H+1}} \quad \text{for } -\tfrac{1}{2} < H \leqslant \tfrac{3}{2} \text{ and } wt \gg 1. \tag{29}$$

Equation (29) shows that the average spectrum of RL FBM obeys a simple power-law behaviour similar to the well known MV FBM but only in the high-frequency domain. With this result, one can envisage that RL FBM may show scale invariance on a small-increments scale or a locally self-affine one. The analysis of the increment property of RL FBM will be presented in a forthcoming paper where we also investigate the fractal characteristics of RL FBM.

5. Conclusion

Time-varying processes with fractional characteristics other than MV FBM have received little interest in the past. RL FBM has been studied in passing by some authors [24, 25] as a simplified version of MV FBM. We note, however, that there exist much difference between the statistical and the spectral properties in these two processes. In this paper we have presented two types of spectrum analysis for RL FBM, namely the double-frequency spectral density and the Wigner–Ville spectrum. The spectral interpretation based on the bifrequency distribution is considered to be informative. It provides a description of the nonstationary behavior in the frequency–frequency plane. However, a physical interpretation of the frequency and energy based on this spectrum is still not fully understood. In fact, the double-frequency spectral density of RL FBM can at best be regarded as 'generalized energy' distribution and the frequencies have to be interpreted in a similar sense. Therefore, a direct physical application of this spectrum is not plausible yet.

The calculation of the Wigner–Ville spectrum of RL FBM gives an exact solution that involves special functions. The time-frequency dependence obtained in the form of the Bessel and Struve functions does not provide a useful and informative picture of the spectrum. The average-power spectrum deduced from the asymptotic Wigner–Ville spectrum showed power-law behavior at high frequencies. This suggests some possible applications of RL FBM to modelling physical phenomena that possess similar asymptotic behaviour.

For example, the asymptotic power-law spectrum obtain above may be applied to modelling irregular time series and fractional noise processes with a $1/f$ spectrum at high frequencies. One such example is the seismic signals generated by certain types of strong ground motions which occur during a random fracture of small-scale fault heterogeneous areas [24]. Another possible application of RL FBM is in the modelling of local random fractal trajectories functions or profiles [26] which can be found in polymer physics [27] and in single-crack trajectories occuring in brittle fractures [28]. Detail studies of these topics are currently in progress.

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